<table>
<tr><th colspan="3">TOPIC PLAN</th></tr>
<tr><td>Partner organization</td><td colspan="2">University ‘Goce Delcev’ - Stip</td></tr>
<tr><td>Topic</td><td colspan="2">Calculus</td></tr>
<tr><td>Lesson title</td><td colspan="2">Optimization problems</td></tr>
<tr><td>Learning objectives</td><td>We use tools from calculus to solve optimization problems. This includes:

- Find first and second derivative of a function
- Find the critical numbers of the function
- Apply first and second derivative test to identify the extrema of the function</td><td rowspan="2">Strategies/Activities
☐Graphic Organizer
☐ Think/Pair/Share
☐Modeling
☐Collaborative learning
☐Discussion questions
☐Project based learning
☐Problem based learning

Assessment for learning
☐Observations
☐Conversations
☐Work sample
☐Conference
☐Check list
☐Diagnostics</td></tr>
<tr><td>Aim of the lecture / Description of the practical problem</td><td>The aim of the lecture is to apply calculus tools in finding the best solution from all feasible solutions. We consider practical problems from geometry, economics and real life applications.

The first practical problem is about maximizing the amount of light going through a window with given shape. In fact, the problem is about maximizing the area of a shape with given constraints.
Problem 1 The upper side of a rectangular window is surmounted by a semicircle (so called Norman window). Thus, the diameter of the semicircle is equal to the width of the rectangle. If the perimeter of the rectangle is 10 m, find the dimensions of the window so that the greatest possible amount of light is admitted.


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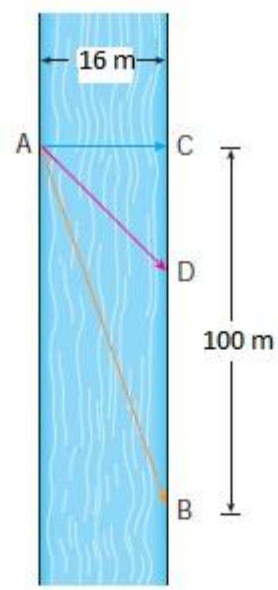
	The second practical problem is about time optimization: we will minimize the time to get from point *A* to point *B.* **Problem 2** A man swims from point *A* on a bank of a straight river, 16 m wide, and wants to reach point *B*, 100 m downstream on the opposite bank, as quickly as possible. He could swim directly across the river to point *C*, and then run to *B*, or he could swim directly to *B*, or he could swim to some point *D* between *C* and *B* and then run to *B*. If he can swim 2 m/sec and run 6 m/sec, where should he land to reach *B* as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man swims.)  	**Assessment as learning** ☐Self-assessment ☐Peer-assessment ☐Presentation ☐Graphic Organizer ☐Homework **Assessment of learning** ☐Test ☐Quiz ☐Presentation ☐Project ☐Published work

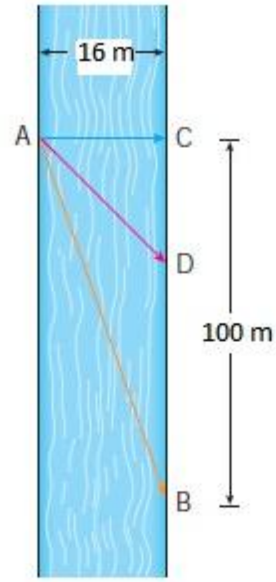
Previous knowledge assumed:	- Critical point of a continuous function - Local and absolute maxima and minima - First and second derivative of a function - Derivative tests - Elementary fact and results from Geometry - Basic facts from kinematics (distance, time, velocity)
Introduction / Theoretical basics	In science, engineering and business one is often interested in problems that involve finding the absolute maximum value or the absolute minimum value of a function. For example, a company is naturally interested in maximizing revenue while minimizing cost. The calculus tools can be used to solve such **optimization problems.** We note that statement of the problem does not usually include the function that is to be optimized. It is our task of first setting up the function that is to be maximized or minimized and then finding its absolute extremum. **Steps in solving Optimization Problems** **1.Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions? **2. Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram. **3. Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized, for example *Q*. Select symbols for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols – for example, *A* for area, *h* for height, *t* for time. We consider first some optimization problems from geometry. **4.** Express *Q* in terms of some of the other symbols from Step 3. **5.** If *Q* has been expressed as a function of more than one variable in Step 4, use the given information to find

	relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one variable in the expression for *Q.* Thus, *Q* will be expressed as a function of *one* variable, say *x,* $Q = f(x)$. Write the domain of this function. **6.** Use the tools from calculus to find the *absolute* maximum or minimum value of *f.* Following these steps, now we are ready to give the solutions of the two practical problems	
Action	**Solution of Problem 1**:  By *x* we denote the radius of the semicircle and by 2*y* its height. We have $y = \frac{5}{2} - x$.The greatest amount of light comes in, if the function of the surface $$f(x) = 4xy + \frac{\pi x^2}{2} = 4x\left(\frac{5}{2} - x\right) + \frac{\pi x^2}{2}$$	

reaches maximum. The derivative is $f'(x)=10-8x+\pi x$, hence the only critical number is $x=\frac{10}{8-\pi}$. Since $f''(x)=-8+\pi<0$, we have a maximum at the critical number $x=\frac{10}{8-\pi}$.

The window has therefore width $\frac{20}{8-\pi}$ = 4.12 m and height $\frac{20-5\pi}{8-\pi}$ = 0.88 m.

Solution of Problem 2:



The swimming distance forms the hypotenuse of a right angled triangle. One of the sides has length 16 m, the other has unknown length *x* = *CD.* Hence, the swimming time is $\sqrt{x^2+16^2}/2$ and the running time is $(100-x)/6$. So, the total time as a function of *x* is

$$T(x)=\frac{\sqrt{x^2+16^2}}{2}+\frac{100-x}{6}.$$

The domain of this function is [0,100]. Notice that if *x* = 0 he swims to *C*, and if *x* = 100 he swims directly to *B*.

The derivative of *T* is

$$T'(x)=\frac{x}{2\sqrt{x^2+16^2}}-\frac{1}{6}.$$

Using the fact that $x\geq 0$, we have

$$T'(x)=0 \quad \Leftrightarrow \quad x=4\sqrt{2}.$$

The only critical number is $x=4\sqrt{2}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain [0,100], we evaluate *T* at all three points:

T(0) = 24,6 *T*(100) = 50,6 $T\left(4\sqrt{2}\right)=24{,}18$.

Since the smallest of these values of *T* occurs when $x=4\sqrt{2}$, the absolute minimum value of *T* must occur there.

We give more examples of optimization problems.

Area and Cost Optimization

Example 1. Maximizing Area

Of all triangles with given perimeter $2s$ and one side with fixed length a, find the triangle with largest area.

Solution. Let us denote the sides of the other two sides of the triangle with a and b. We know that by Heron's formula its area *A* is given by

$$A=\sqrt{s(s-a)(s-b)(s-c)}\,.$$

We put $b=x$, then $2s=a+x+c$ and therefore $s-c=a+x-s$. Thus we can write the area *A* in the form

$$Q=f(x)\ \ A=\sqrt{s(s-a)(s-x)(a+x-s)}\,.$$

This function has a maximum in the same points where the function

$$f(x)=(s-x)(a+x-s)$$

has a maximum. It holds

$$(s-x)+(a+x-s)=a\,,$$

and we know that the product of two numbers with fixed sum is the largest when they are equal. Indeed, the function

$$g(x)=x(a-x)$$

has as the only critical point the zero $x=a/2$ of the derivative

$$g'(x)=a-2x\,.$$

Since $g''(x)=-2<0$, the function $g(x)=x(a-x)$ has maximum at $x=a/2$.

Hence, in our case, the maximum for $f(x)=(s-x)(a+x-s)$ is at

$$s-x=a+x-s=a/2\,,$$

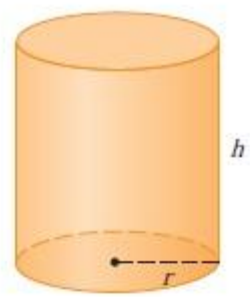
and this is $b=c$. We conclude that the solution is the isosceles triangle. From all triangles with given perimeter and one fixed side, the isosceles triangle has the maximal area.

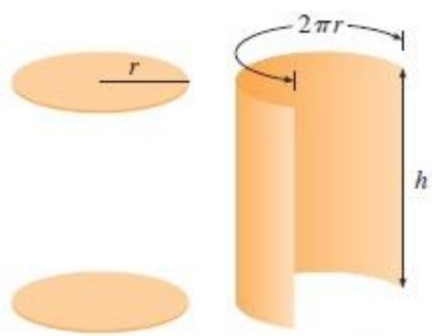
Next example is from economics where a certain product is produced in a shape so that the production cost is minimized.

Example 2. Minimizing Area or Cost Minimization

A juice can is to be made in the form of a right circular cylinder and have a volume *V*. Find the dimensions of the can so that the least amount of material is used in its construction.

Solution. Material = total surface area of can
= area of top + area of bottom + area of lateral side.





Area $2(\pi r^2)$ Area $(2\pi r)h$

Let *r* be the radius and *h* the height of the can, see diagram. So the surface area is

$$A = 2\pi r^2 + 2\pi rh.$$

We use the fact that $V = \pi r^2 h$, which gives $h = V/\pi r^2$. Eliminating *h* from the surface area gives

$$A = 2\pi r^2 + \frac{2V}{r}.$$

Therefore, the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2V}{r} \qquad r > 0.$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2V}{r^2} = \frac{2(2\pi r^3 - V)}{r^2}.$$

From $A'(r) = 0$, we find that the only critical number is

	$$r=\sqrt[3]{V/2\pi}\ .$$ We observe that the second derivative $A''(r)=4\pi+\dfrac{4V}{r^3}>0$ for positive *r* > 0. Hence, the critical number $r=\sqrt[3]{V/2\pi}$ gives rise to an absolute minimum. The value of *h* corresponding to $r=\sqrt[3]{V/2\pi}$ is $$h=V/\pi r^2=\sqrt[3]{4V/\pi}=2r\ .$$ Thus, to minimize the amount of material and so to minimize the cost of the can with volume *V*, the radius should be $r=\sqrt[3]{V/2\pi}$ and the height of the can should be equal to twice the radius, namely, the diameter. Question for the students in the class: ➢ What is the unknown function in the optimization problem? ➢ How do you connect the unknown variables? ➢ What is the derivative and the critical number of the function? ➢ What is the minimum/maximum in the optimization problem?	

Materials / equipment / digital tools / software	We use the textbooks from the references. For equipment in the classroom we need the usual black board and chalks. Digital tools: laptop, projector, smart board. The practical problems can be solved without using software – calculators are enough.	

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<tr><td>Consolidation</td><td colspan="2">Use of materials, equipment, digital tools, software by teachers and students;
The teacher's discussion with the students through appropriate questions;
Independent solving of simple tasks by the students under the supervision of the teacher;
Given of examples by the teacher for introducing a new concept in a cooperation and a discussion with the students;
Assignment of homework by the teacher with a time limit until the next class</td></tr>
<tr><td colspan="3">Reflections and next steps</td></tr>
<tr><td colspan="2">Activities that worked</td><td>Parts to be revisited</td></tr>
<tr><td colspan="2">After the class, the teacher according to his personal perceptions regarding the success of the class fills in this part.</td><td>Through the success of the homework done by the students, questions and discussion at the beginning of the next class, the teacher comes to the conclusion which parts of this class should be revised.</td></tr>
<tr><td colspan="3">References</td></tr>
<tr><td colspan="3">J. Stewart: Calculus – Early Transcendentals, Thomson 2008
M. Lukarevski: Mathematics for computer scientists (in Macedonian), Univ. ‘Goce Delcev’ – Stip, 2019</td></tr>
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